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A New Four Parameter Kumaraswamy Distribution

Osowole O. I.[1], Izunobi Chinyeaka H.[2] and Salisu, S. U.[3]

[1]Reader, Department of Statistics, University of Ibadan, Nigeria
academicprofessor2013@gmail.com

[2]Lecturer II, Department of Statistics, Federal University of Technology, Imo State, Nigeria
hostensia.izunobi@futo.edu.ng

[3]Lecturer, Department of Mathematics and Statistics, Kaduna Polytechnic, Kaduna State, Nigeria
umarss83@yahoo.com

Abstract

This study considered a new addition to the Kumaraswamy generated family of distributions. The new distribution is referred to as the Kumaraswamy Mukherjee-Islam distribution. Some theoretical properties of the new distribution were derived under certain conditions. The new distribution is therefore expected to offer more flexibility than the baseline Mukherjee-Islam distribution.

Keywords: *Kumaraswamy Mukherjee-Islam distribution, Mukherjee-Islam distribution, Generalized distributions, Statistics, Applied mathematics, Exponentiation, Moments.*

1.0 Introduction

Generalized distributions are needful in Statistics, Applied Mathematics and allied disciplines because it allows inference to be made on sub distributions embedded in the generalized distributions. This is often desired to be able to demonstrate the flexibility of these generalized distributions.

Generalized distributions are generally obtained from selected baseline (parent) distributions using different approaches. Some of these approaches include exponentiation, transmutation, quasi-transmutation, weighting of distributions, extension, convolution, power transformation, composition and generalizations from certain generalized family of distributions. Examples of these generalized families of distributions are Kumaraswamy generated family of distributions, Weibull generated family of distributions, Marshall-Olkin generated family of distributions, Beta generated family of distributions and Generalized exponential generated family of distributions. The following are examples of some generalized distributions in literature: Kumaraswamy-Dagum distribution by [1]; Transmuted linear exponential distribution by [2]; Slashed exponentiated rayleigh distribution by [3]; Slashed rayleigh distribution by [4]; Exponentiated generalized extended exponential distribution by [5]; Marshall-Olkin Extended Burr III distribution by [6]; Alpha Power weibull distribution by [7]; Extended generalized exponential distribution by [8]; Length Biased Sushila distribution by [9]; Length Biased Weighted Generalized Uniform distribution [10]; Size Biased distribution by [11]; Beta exponentiated modified weibull distribution by [12]; Generalized length biased exponential distribution by [13]; Exponentiated generalized exponential Dagum distribution by [14]; Modified Shukla Distribution by [15] and k-Modified Generalized Uniform distribution by [16].

Based on the foregoing, this study is therefore an attempt to contribute to the existing Kumaraswamy generated family of distribution by proposing the theoretical exploration of the Kumaraswamy Mukherjee-Islam Distribution (KMID). It is hoped that this new addition will demonstrate enhanced flexibility and superiority over the existing baseline (parent) distribution.

2.0 The Kumaraswamy Mukherjee-Islam Distribution

The Kumaraswamy generated family of distributions based on the Kumaraswamy distribution will be chosen to generate the new generalized distribution based on the observation of [17] who noted that the ‘‘distribution can be used to model many random processes and uncertainties’’ (p.741). According to [18] and [19], the Kumaraswamy generated family of distributions has cumulative distribution function (cdf) and probability density function (pdf) given as

$$G(x) = 1 - [1 - (F(x))^c]^d; c, d > 1 \qquad \text{...(1.0)}$$

From (1.0), the probability density function (pdf) is given as

$$g(x) = cdf(x)[F(x)]^{b-1}[1 - (F(x))^c]^{d-1}; c, d > 1 \qquad \text{...(2.0)}$$

where c and d are additional shape parameters. Also, F(x) and f(x) are the baseline (parent) cdf and pdf of the selected parent distribution; that is the Mukherjee-Islam Distribution with cdf and pdf given as

$$F(x) = \left[\left(\frac{x}{a}\right)\right]^b, \ 0 < x < a; \ \text{a}, b > 0 \qquad \text{...(3.0)}$$

$$f(x) = \frac{b}{a^b} x^{b-1}, \ 0 \le x \le a; \ \text{a}, b > 0 \qquad \text{...(4.0)}$$

where a is a scale parameter and b is a shape parameter

Using (3.0) and (4.0) in (1.0) and (2.0), we have the Kumaraswamy Mukherjee-Islam Distribution (KMID) with the cdf and pdf given in (5.0) and (6.0) as

$$G_{KMID}(x) = 1 - [1 - \left(\frac{x}{a}\right)^{bc}]^d; 0 < x < a, a > 0, b, c, d > 1 \qquad \text{...(5.0)}$$

$$g_{KMID}(x) = cd \frac{b}{a^b} x^{b-1} \left[\left(\frac{x}{a}\right)^b\right]^{c-1} \left[1 - \left(\frac{x}{a}\right)^{bc}\right]^{d-1}; 0 < x < a, a > 0, b, c, d > 1 \qquad \text{...(6.0)}$$

The Mukherjee-Islam Distribution was selected as the baseline distribution as per the observation of [20] who observed that ‘‘the mathematical form of the distribution is simple and can be handled without much complications'' (p.357). The plots of the cdf and pdf of the Kumaraswamy Mukherjee-Islam Distribution (KMID) are shown below in Figures (1.0) and (2.0).

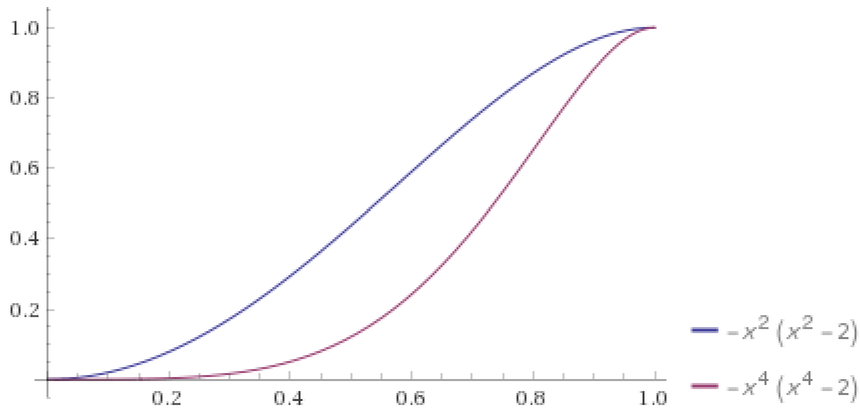

Figure (1.0): The cdf plot of the KMID at different values of the parameters for 0 < x <1

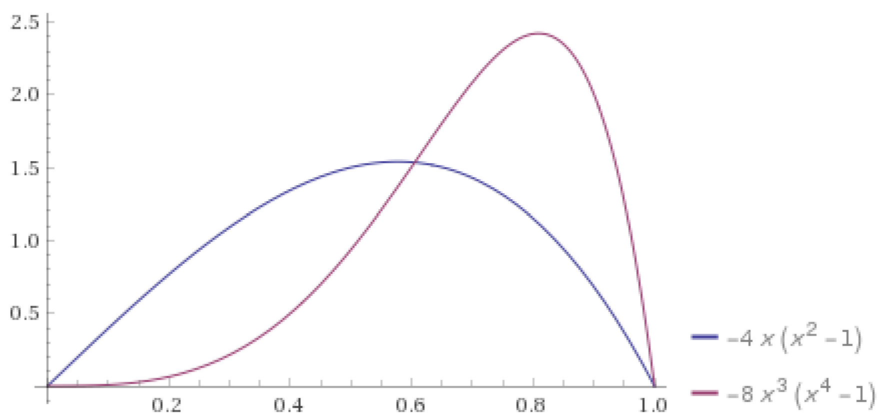

Figure (2.0): The pdf plot of the KMID at different values of the parameters for 0 < x <1

The two plots above reveal that the shapes of the plots vary with changes in the values of the parameters. The pdf plot specifically reveals that the KMID is uni-modal.

2.1 Theoretical Characterization of the Kumaraswamy Mukherjee-Islam Distribution

2.1.1 Moments

The k^{th} moment about zero for a random variable X from the Kumaraswamy Mukherjee-Islam Distribution is

$$\mu_k^1 = E(X^K) = \int_0^a x^k \; g_{KMID}(x)\, dx$$
$$= \int_0^a x^k cd\left(\frac{b}{a^b} x^{b-1}\right)\left[\left(\frac{x}{a}\right)^b\right]^{c-1}\left[1-\left(\frac{x}{a}\right)^{bc}\right]^{d-1} dx$$
$$= a^{k+bc-b(c-1)-b} d\mathrm{B}(d, \frac{k+b(c-1)+b}{bc}), k \ge 1$$

where $\mathrm{B}(i, j) = \dfrac{\Gamma(i)\Gamma(j)}{\Gamma(i+j)}$

To simplify the k^th moment about zero, μ_k', we shall let c = d =2 in

$$\mu_k^1 = E(X^K) = \int_0^a x^k cd\left(\frac{b}{a^b} x^{b-1}\right)\left[\left(\frac{x}{a}\right)^b\right]^{c-1}\left[1-\left(\frac{x}{a}\right)^{bc}\right]^{d-1} dx$$

so that

$$\mu_k^1 = E(X^K) = \int_0^a x^k cd\left(\frac{b}{a^b} x^{b-1}\right)\left[\left(\frac{x}{a}\right)^b\right]^{c-1}\left[1-\left(\frac{x}{a}\right)^{bc}\right]^{d-1} dx$$
$$= \int_0^a 4x^k\left(\frac{b}{a^b} x^{b-1}\right)\left[\left(\frac{x}{a}\right)^b\right]\left[1-\left(\frac{x}{a}\right)^{2b}\right] dx$$
$$= \frac{8a^k b^2}{(k+2b)(k+4b)} \forall c = d = 2$$

Specifically,

$$\mu_1^1 = \mathrm{E}(X) = mean = \frac{8ab^2}{(2b+1)(4b+1)} \forall c = d = 2$$
$$\mu_2^1 = \mathrm{E}(X^2) = \frac{2a^2b^2}{(b+1)(2b+1)} \forall c = d = 2$$

Combining the results yields the variance of X, V(X). That is,

$$Variance\,(X) = \sigma_X^2 = \mathrm{E}(X^2) - \left[\mathrm{E}(X)\right]^2$$
$$= \frac{2a^2b^2}{(2b+1)(b+1)} - \left(\frac{8ab^2}{(2b+1)(4b+1)}\right)^2$$
$$= \frac{2a^2b^2(10b+1)}{(b+1)(2b+1)^2(4b+1)^2}\ \forall c = d = 2$$

Next, we obtain the Harmonic Mean. The Harmonic Mean ($\mathrm{E}(\frac{1}{X})$) for a random variable X from the Kumaraswamy Mukherjee-Islam Distribution is

$$H.M(X) = E(\frac{1}{X}) = \int_0^a \frac{1}{x}\left(\frac{4b}{a^b}x^{b-1}\right)\left[\left(\frac{x}{a}\right)^b\right]\left[1-\left(\frac{x}{a}\right)^{2b}\right]dx$$
$$= \frac{8b^2}{a(2b-1)(4b-1)}\ \forall c = d = 2$$

2.1.2 Moment Generating Function

The moment generating function (m.g.f.) of a random variable X from the Kumaraswamy Mukherjee-Islam distribution is defined as

$$M_X(t) = \mathrm{E}(e^{tX}) = \int_0^a e^{tX} g_{KMID}(x)dx$$
$$= \int_0^a \left[1 + tX + \frac{(tX)^2}{2!} + \ldots\right] g_{KMID}(x)dx$$
$$= \int_0^a \sum_{j=0}^{\infty} \frac{t^j}{j!} X^j\, g_{KMID}(x)dx$$
$$= \sum_{j=0}^{\infty} \frac{t^j}{j!}\mu_j' = \sum_{j=0}^{\infty} \frac{t^j}{j!}\left(\frac{8a^jb^2}{(j+2b)(j+4b)}\right) \forall c = d = 2$$

2.1.3 Characteristic Function

The characteristic function (c. f.) of a random variable X from the Kumaraswamy Mukherjee-Islam distribution is defined as

$$\begin{aligned}\phi_x(t) &= M_X(it)\\ &= \mathrm{E}(e^{itX})\\ &= \int_0^{\frac{1}{c}} e^{itX} g_{KMID}(x)dx\\ &= \sum_{j=0}^{\infty}\frac{(it)^j}{j!}\mu_j' = \sum_{j=0}^{\infty}\frac{(it)^j}{j!}\left(\frac{8a^jb^2}{(j+2b)(j+4b)}\right)\forall c=d=2\end{aligned}$$

2.1.4 Cumulant Generating Function

The cumulant generating function (c. g. f.) of a random variable X from the Kumaraswamy Mukherjee-Islam Distribution is defined as

$$\begin{aligned}K_X(t) &= In\left[M_X(t)\right]\\ &= In\left[\sum_{j=o}^{\infty}\frac{t^j}{j!}\mu_j^1\right]\\ &= In\left[\sum_{j=0}^{\infty}\frac{t^j}{j!}\left(\frac{8a^jb^2}{(j+2b)(j+4b)}\right)\right]\forall c=d=2\end{aligned}$$

2.1.5 Coefficients of Variation and Dispersion

The coefficients of variation and dispersion for the Kumaraswamy Mukherjee-Islam Distribution are defined as

$CV=\dfrac{\sigma}{E(X)}\ and\ CD=\dfrac{\sigma^2}{E(X)}$. Specifically,

$$CV=\frac{(2b+1)(4b+1)\sqrt{\dfrac{2a^2b^2(10b+1)}{(b+1)(2b+1)^2(4b+1)^2}}}{8ab^2}\ \ \forall c=d=2$$

and $$CD=\frac{2a^2b^2(10b+1)(2b+1)(4b+1)}{8ab^2(b+1)(2b+1)^2(4b+1)^2}\forall c=d=2$$

2.1.6 Hazard Function

The hazard function (h. f.) of a random variable X from the Kumaraswamy Mukherjee-Islam Distribution is defined as

$$h_{KMID}(x) = \frac{g_{KMID}(x)}{1-G_{KMID}(x)}$$

$$= \frac{\frac{4b}{a^b} x^{b-1}\left[\left(\frac{x}{a}\right)^b\right]\left[1-\left(\frac{x}{a}\right)^{2b}\right]}{\left(1-\left[\frac{x}{a}\right]^{2b}\right)^2} \forall c = d = 2$$

2.1.7 Reserve Hazard Function

The reserve hazard function (r. h. f.) of a random variable X from the Kumaraswamy Mukherjee-Islam Distribution is defined as

$$h_{r,KMID}(x) = \frac{g_{KMID}(x)}{G_{KMID}(x)}$$

$$= \frac{\frac{4b}{a^b} x^{b-1}\left[\left(\frac{x}{a}\right)^b\right]\left[1-\left(\frac{x}{a}\right)^{2b}\right]}{1-[1-\left[\left(\frac{x}{a}\right)^{2b}\right]^2}; \forall c = d = 2$$

2.1.8 Mills Ratio

The Mills Ratio of the Kumaraswamy Mukherjee-Islam Distribution is defined as

$$\left(\frac{1}{reserve\ hazard\ function}\right)=\left(\frac{1-[1-\left[\left(\frac{x}{a}\right)^{2b}\right]^{2}}{\frac{4b}{a^{b}}x^{b-1}\left[\left(\frac{x}{a}\right)^{b}\right]\left[1-\left(\frac{x}{a}\right)^{2b}\right]}\right)\forall c=d=2$$

2.1.9 Survival Function

The survival function (s.f.) of a random variable X from the Kumaraswamy Mukherjee-Islam distribution is defined as

$$S_{X,KMID}(x)=1-G_{KMID}(x)$$
$$=[1-\left[\left(\frac{x}{a}\right)^{2b}\right]]^{2}\forall c=d=2$$

2.1.10 Order Statistics

Suppose $X_{(1)}, X_{(2)},\ldots, X_{(n)}$ are the order statistics from the random sample $X_1, X_2, \ldots, X_n$ from the Kumaraswamy Mukherjee-Islam Distribution with $g_{KMID}(x)$ and $G_{KMID}(x)$ as the pdf and cdf, the pdf of the r^{th} order statistic where $1 \le r \le n$ is given as

$$h_{(r)}(x) = \frac{n!}{(r-1)!(n-r)!}\, g_{KMID}(x)\left[G_{KMID}(x)\right]^{r-1}\left[1-G_{KMID}(x)\right]^{n-r}$$

$$=\frac{n!}{(r-1)!(n-r)!}\left[\frac{4b}{a^{b}}x^{b-1}\left[\left(\frac{x}{a}\right)^{b}\right]\left[1-\left(\frac{x}{a}\right)^{2b}\right]\right]\left(1-[1-\left[\left(\frac{x}{a}\right)^{2b}\right]^{2}\right)^{r-1}\left([1-\left[\left(\frac{x}{a}\right)^{2b}\right]]^{2}\right)^{n-r}\forall c=d=2$$

By setting r = n and r =1 in the function ($h_{(r)}(x)$) above, we have the distributions for the largest and lowest order statistics. For the largest order statistic, we have that

$$h_{(r=n)}(x) = n\left[\frac{4b}{a^b}x^{b\text{-}1}\left[\left(\frac{x}{a}\right)^b\right]\left[1\text{-}\left(\frac{x}{a}\right)^{2b}\right]\right]\left(1-[1-\left[\left(\frac{x}{a}\right)^{2b}\right]^2\right)^{n\text{-}1} \quad \forall c = d = 2$$

For the lowest order statistic, we have that

$$h_{(r=1)}(x) = n\left[\frac{4b}{a^b}x^{b\text{-}1}\left[\left(\frac{x}{a}\right)^b\right]\left[1\text{-}\left(\frac{x}{a}\right)^{2b}\right]\right]\left([1-\left[\left(\frac{x}{a}\right)^{2b}\right]]^2\right)^{n\text{-}1} \quad \forall c = d = 2$$

2.1.11 Random Number Generation

By the quantile function (i.e the inverse of the cdf), random numbers can be generated for the Kumaraswamy Mukherjee-Islam distribution as follows: Let $1\text{-}[1\text{-}(\frac{x}{a})^{2b}] = u$ where U is a random variable from the Uniform (0, 1) distribution. The random number generation will be done by solving the equation for x. After some algebraic manipulations, we have that
$x = e^{\frac{In\,u}{2b}+In\,a}$. The equation for the random number generation was obtained by choosing c = 2 and d =1 in the quantile function.

3.0 Conclusion

The Kumaraswamy Mukherjee-Islam Distribution has been derived successfully in this study as well as some of its essential theoretical properties. The distribution was derived as a generalized distribution from the Kumaraswamy generated family of distributions with the Mukherjee-Islam distribution as the baseline distribution. The Kumaraswamy Mukherjee-Islam Distribution has variability lower than that of the baseline distribution and this is an indication of improved flexibility and superiority.

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